# MPC with one free control action for constrained LPV systems

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Abstract—Based on the assumption that the parameter can be measured in real time, we propose a model predictive control (MPC) method for linear-parameter varying (LPV) systems subject to possibly asymmetric constraints which adopts the analogous framework of terminal control law, terminal set and terminal penalty of nonlinear model predictive control. The optimization problem is formulated as a convex optimization problem and, recursive feasibility and closed-loop stability are guaranteed by its feasibility at initial time. For LPV systems with symmetric constraints, we reformulate the convex optimization problem as a semi-definite program. Numerical examples demonstrate the properties of the proposed MPC design.

## I. INTRODUCTION

Model predictive control or receding horizon control is a class of optimization based control methods in which a control sequence is determined by optimizing a finite horizon cost at each sampling instant, based on an explicit process model and state measurements. The first control action of the optimal sequence is applied to the plant. At the next sampling instant, the optimization problem is solved again with new measurements, and the control input is updated. Due to its ability to handle constraints on inputs and states, the method has received much interest in both academic community and industrial society over the last 30 years, see e.g. [1,2].

Linear-parameter varying systems are linear systems whose dynamics depend on time-varying parameters, which takes its values in pre-specified set. The analysis and controller synthesis of LPV systems play an important role in control theory and application since both nonlinear systems and linear systems with model perturbation can be dealt with [3, 4]. Predictive control of LPV sytems has been effected through the concept of ellipsoidal invariant sets [5] which has been used to design a state-feedback control law that minimizes an upper bound on the "worst-case" infinite horizon objective function, and keeps the system state inside an invariant feasible set. Many results in the literature represent extensions of [5], for example, schemes with enlarged feasible region and reduced computational burden have been developed. Using parameterdependent Lyapunov functions, [6-8] propose proce-

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dures which do not require the quadratic stabilizability of the given system. An improved approach is proposed in [9] which deploys a fixed state-feedback law with perturbations. The algorithm requires a modest amount of online computation and introduces extra degree of freedom to enlarge the volume of the relevant invariant set. A parameter-dependent control law in the framework of gain-scheduling is proposed in [10] which offers potential performance improvements over a fixed control law. A self-scheduling controller is adopted in [11] which reduces conservativeness and improves feasibility characteristics at the cost of online heavily computational burden. Robust receding horizon control schemes are proposed by [12, 13] which are based on the minimization of the worst-case stage cost with a finite terminal weighting cost. Only symmetric constraints are considered in the aforementioned works which restrict the potential application of the proposed schemes.

This paper provides a general framework to design a model predictive controller for LPV systems with possibly asymmetric constraints. The online optimization problem, which is formulated as a convex optimization problem, allows the first control action to be chosen freely, while the succeeding control actions can be determined by an offline calculated terminal control law. Feasibility and closed-loop stability are guaranteed by feasibility of the optimization problem at initial time. For LPV systems with symmetric constraints, the convex optimization problem can be formulated as a semi-definite program (SDP).

The remainder of this paper is organized as follows. Section 2 presents the formulation of the optimization problem that will be solved online. Feasibility of the proposed optimization problem, stability of the closedloop system and an improved algorithm for system with symmetric constraints are discussed in detail in Section 3. Numerical examples to illustrate the effectiveness of the algorithm are given in Section 4. Section 5 concludes the paper with a brief summary.

#### II. PROBLEM FORMULATION

Consider discrete-time linear parameter-varying systems of the form

$$x(k+1) = A(\theta(k))x(k) + B(\theta(k))u(k), \quad (1a)$$

$$y(k) = C(\theta(k))x(k) + D(\theta(k))u(k), \quad (1b)$$

subject to the constraints

$$y(k) \in \mathcal{H},\tag{2}$$

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where  $\mathcal{H}$  is a compact set which contains the origin in its interior. Let  $x(k) \in \mathbb{R}^{n_x}$  denote the state,  $u(k) \in \mathbb{R}^{n_u}$  the control input, and  $y(k) \in \mathbb{R}^{n_y}$  the constraints output which is not necessarily measurable.

If the constraint sets are symmetric, we say the system has symmetric constraints. Otherwise, the system has asymmetric constraints. If the constraints are symmetric we write them in element-wise fashion as

$$\mathcal{H}_0 := \{ y(k) \in R^{n_y} | y_{m,max} \le y_m(k) \\ \le -y_{m,max}, m = 1, \dots, n_y \},$$
 (3)

where  $y_{max}$  is a given constant compatible vector. The system matrices  $A(\theta_k) \in \mathbb{R}^{n_x \times n_x}$ ,  $B(\theta(k)) \in \mathbb{R}^{n_y \times n_u}$ ,  $C(\theta(k)) \in \mathbb{R}^{n_y \times n_x}$  and  $D(\theta(k)) \in \mathbb{R}^{n_y \times n_u}$ are assumed to depend on the parameter vector  $\theta(k) := [\theta_1(k), \theta_2(k), \cdots, \theta_N(k)]^T \in \mathbb{R}^N$ , which belongs to a convex polytope  $\mathcal{P}$ ,

$$\mathcal{P}: \quad \sum_{j=1}^{N} \theta_j(k) = 1, \quad 0 \le \theta_j(k) \le 1.$$

Clearly, as  $\theta(k)$  varies inside the polytope  $\mathcal{P}$ , the matrices of system (1) vary inside a corresponding polytope  $\Xi$ , i.e.

$$\begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix} \in \Xi,$$
(4)

which is defined by the convex hull of N local extremal matrices  $[A_i, B_i, C_i, D_i], i = 1, 2, \dots, N$ ,

$$\Xi := Co\left\{ \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix}, \begin{bmatrix} A_2 & B_2 \\ C_2 & D_2 \end{bmatrix}, \dots, \begin{bmatrix} A_N & B_N \\ C_N & D_N \end{bmatrix} \right\}.$$

Therefore, we can write the matrices of system (1) as

$$A(\theta(k)) = \sum_{j=1}^{N} \theta_j(k) A_j, \ B(\theta(k)) = \sum_{j=1}^{N} \theta_j(k) B_j,$$
$$C(\theta(k)) = \sum_{j=1}^{N} \theta_j(k) C_j, \ D(\theta(k)) = \sum_{j=1}^{N} \theta_j(k) D_j.$$

Assume that no mismatch exists between the model and the plant, and both the parameter vector  $\theta(k)$  and the state x(k) are available in real-time. Thus, at every sampling time k, the current system matrices  $\begin{bmatrix} A(\theta(k)) & B(\theta(k)) \\ C(\theta(k)) & D(\theta(k)) \end{bmatrix}$  are known exactly but the future system matrices  $\begin{bmatrix} A(\theta(k+i)) & B(\theta(k+i)) \\ C(\theta(k+i)) & D(\theta(k+i)) \end{bmatrix}$ ,  $i \ge 1$  are uncertain and vary inside the prescribed polytope.

In this paper, we first consider the infinite horizon MPC problem [5]:

**Problem P1:** Consider the system (1)-(3). For a given initial state x(k), find a control law  $u = \kappa(x)$  such that

$$J_{\infty}(k) = \max_{\theta(k+i) \in \mathcal{P}, i \ge 1} \sum_{i=0}^{\infty} \left\{ x(k+i/k)^T \\ Qx(k+i/k) + u(k+i/k)^T Ru(k+i/k) \right\},$$
(5)

is minimized with respect to the time-varying parameter  $\theta(k)$  and output constraints (2), where  $Q \in R^{n_x \times n_x}$ and  $R \in R^{n_u \times n_u}$  are positive definite weighting matrices, and x(k + i/k) and u(k + i/k) denote the future state and input predicted at time instant k.

**Definition 1:** A set  $\Omega$  is an invariant set if  $x(k) \in \Omega$ implies  $x(k+i) \in \Omega$  for  $\forall i \geq 0$ .

## III. MPC WITH ONE FREE CONTROL ACTION

In this section we first reformulate the prescribed optimization problem P1 based on the concept of invariant sets and propose a novel MPC scheme with one free control action. Stability of the closed-loop system and feasibility of the related optimization problem can be guaranteed by the feasibility of the optimization problem at initial time. For the case of symmetric constraints, the convex optimization problem is formulated as an SDP.

## A. MPC for LPV systems

Define  $\Omega(\alpha)$  as a neighborhood of the origin

$$\Omega(\alpha) := \{ x \in \mathbb{R}^{n_x} | V(x) \le \alpha, \alpha > 0 \}.$$
(6)

Then,  $\Omega(\alpha)$  is a level set of the positive definite function  $V(x) := x^T P x$ ,  $P \succ 0$ . The following assumption is needed to reformulate the infinite horizon cost (5).

Assumption 1: Suppose that there exists a continuous local controller  $u = \kappa(x)$  such that the following conditions are satisfied,

B0)  $y(k) \in \mathcal{H}$ , for all  $x(k) \in \Omega(\alpha)$ , B1) V(x) satisfies inequality

$$x(k+i)^{T}Qx(k+i) + \kappa (x(k+i))^{T}R\kappa (x(k+i)) + V(x(k+i+1)) - V(x(k+i)) \le 0, \ i \ge 0,$$

$$(7)$$

for all  $\theta \in \mathcal{P}$  and all  $x(k+i) \in \Omega(\alpha)$ .

According to Assumption 1,  $\Omega(\alpha)$  has the following properties:

- The origin is contained in the interior of Ω(α), due to V(x) > 0, ∀x ∈ Ω, and α > 0,
- Ω(α) is closed and connected, due to the continuity of V in x,
- Since (7) holds, Ω(α) is a robustly positively invariant set for the "unknown" parameter of the LPV system (1) controlled by u = κ(x).

The set  $\Omega(\alpha)$  and the function V(x) can be applied as terminal set and terminal penalty of the considered system [1,14].

**Remark 1:** The terminal set in [15] is a neighborhood of the equilibrium which satisfies constraints and positive invariance. Here,  $\Omega(\alpha)$  is "robustly" positively invariant and constraints are satisfied for all admissible and "unknown" parameters.

**Lemma 1:** Suppose Assumption 1 is satisfied, then V(x(k + 1/k)) is an upper bound on  $\max_{\theta(k+i)\in\mathcal{P}}\sum_{i=1}^{\infty} \left\{ x(k+i/k)^T Q x(k+i/k) + \kappa \left( x(k+i/k) \right)^T R \kappa \left( x(k+i/k) \right) \right\}.$ Proof: Summing and the set of the

*Proof:* Summing up the inequality (7) from k = 1 to  $\infty$ , yields

$$V(x(\infty/k)) - V(x(k+1/k)) \leq -\max_{\theta(k+i)\in\mathcal{P}} \sum_{i=1}^{\infty} x(k+i/k)^T Q x(k+i/k) + \kappa (x(k+i/k))^T R \kappa (x(k+i/k)),$$
(8)

Since  $Q \succ 0$  and  $R \succ 0$ , it follows from (7) that

$$V(x(k+i+1/k)) - V(x(k+i/k)) \le 0, \forall i \ge 0$$

Thus,  $V(\cdot)$  is a Lyapunov function and therefore we have  $V(x(\infty/k)) = 0$ . Plugging this into (8) proofs the lemma.

With the results of Lemma 1 we approximate the infinite horizon cost as

$$J(k) := x(k/k)^T Q x(k/k) + u(k/k)^T R u(k/k) + x(k+1/k)^T P x(k+1/k).$$
(9)

Therefore, problem P1 for the current state x(k) and the current parameter  $\theta(k)$  is reformulated as follows:

**Problem P2:** At time k, consider the optimization problem

$$\underset{\bar{u}(k/k)}{\text{minimize}} \quad J(k) \tag{10a}$$

subject to

$$\begin{split} \bar{x}(k+1/k) &= A(\theta(k))\bar{x}(k/k) + B(\theta(k))\bar{u}(k/k),\\ \bar{y}(k/k) &= C(\theta(k))\bar{x}(k/k) + D(\theta(k))\bar{u}(k/k),\\ \bar{y}(k/k) &\in \mathcal{H}, \quad \bar{x}(k/k) = x(k),\\ \bar{x}(k+1/k) &\in \Omega(\alpha), \end{split}$$

where  $\Omega(\alpha)$  is the terminal set, and  $x(k + 1/k)^T P x(k+1/k)$  is the terminal penalty.

In P2, the first control action can be chosen freely, while the future inputs can be chosen according to the feedback control law of Assumption 1.

**Remark 2:** If  $\mathcal{H}$  is a convex set, then the optimization problem P2 is a convex optimization problem [16].

**Remark 3:** If not only the parameter but also its rate of variation, although not known a priori, are online available, the prediction horizon in (9) can be chosen as N = 2, i.e.  $J(k) := \sum_{i=0}^{1} \left\{ x(k + i/k)^T Q x(k + i/k) + u(k + i/k)^T R u(k + i/k) \right\} + x(k + 2/k)^T P x(k + 2/k)$ , since also x(k + 2/k) can be calculated exactly at time k.

**Remark 4:** Furthermore, we can choose N arbitrarily large if the varying parameter  $\theta$  is a function of state  $x(\cdot)$  and control  $u(\cdot)$ . In this case  $\theta(k+i/k)$  can be determined by  $x(\cdot)$  and  $u(\cdot)$ , and x(k+i+1/k) can be predicted by  $\theta(k+i/k)$  and  $u(\cdot)$  in a recursive way. Unfortunately, this will introduce nonlinear terms into the optimization problem which will lead to the optimization problem P2 being nonconvex.

According to the principle of MPC, the open loop optimization problem P2 is solved repeatedly at each

time instant k based on the measurement x(k) and  $\theta(k)$ . The following theorem investigates the feasibility and stability of system (1) with the proposed model predictive controller.

Theorem 1: Suppose that

- (a) for the LPV system (1), there exists a locally asymptotically stabilizing controller u = κ(x), a continuously differentiable, positive definite function V(x) = x<sup>T</sup>Px that locally satisfies (7), and a positively invariant set Ω(α) defined by (6),
- (b) the open-loop optimal control problem P2 is feasible at time k = 0.

Then,

- (i). the optimal control problem P2 is feasible for all k > 0,
- (ii). the closed-loop system is nominally asymptotically stable with the region of attraction *D* being the set of all states for which the open-loop optimal control problem has a feasible solution.

*Proof:* In what follows  $u^*(k)$ ,  $J^*(k)$  denote the optimal solution and the optimal value of the optimization problem solved at sampling time instant k.  $x^*(k+1/k)$  denotes the optimal predicted evolution of the system, i.e.  $x^*(k+1/k) = A(\theta(k))x(k) + B(\theta(k))u^*(k)$ .

Part (i). Since, by assumption, there are no disturbances and mismatch, the state measurement at time k + 1 is  $x(k + 1) = x^*(k + 1/k)$ . In virtue of the optimization problem P2,  $x^*(k + 1/k) \in \Omega(\alpha)$ , i.e.  $x(k + 1) \in \Omega(\alpha)$ . It follows from Assumption 1 that  $u(k + 1) = \kappa(x(k + 1))$  is a feasible solution to the optimization problem P2 at time k + 1 for all  $x(k + 1) \in \Omega$ , which proofs part (i) of the Theorem.

Part (ii). Consider the feasible solution at time k+1 obtained in Part (i). Following standard steps in the stability proofs of MPC [1, 15], we obtain

$$J(k+1) = x^{T}(k+1/k+1)Qx(k+1/k+1) +u^{T}(k+1/k+1)Ru(k+1/k+1) +x^{T}(k+2/k+1)Px(k+2/k+1) = J^{*}(k)-x^{T}(k/k)Qx(k/k)-u^{T}(k/k)Ru(k/k) +x^{T}(k+2/k+1)Px(k+2/k+1) + x^{T}(k+1)Px(k+1)+x^{T}(k+1)Qx(k+1) +\kappa^{T}(x(k+1))R\kappa(x(k+1)),$$

with x(k + 1/k) = x(k + 1/k + 1) = x(k + 1) and  $u(k + 1/k + 1) = \kappa(x(k + 1))$ . In virtue of (7), this implies that

$$J^{*}(k+1) \leq J(k+1) \leq J^{*}(k) - x(k)^{T}Qx(k) -u^{*}(k)^{T}Ru^{*}(k).$$
(12)

Due to  $Q \succ 0$  and  $R \succ 0$ , and its nonincreasing evolution, we infer that  $\lim_{k\to\infty} x(k) = 0$ . Furthermore, x = 0 is asymptotically stable due to the continuity of the value function J(k) in x and u.

## B. Offline calculation

The online optimization problem P2 requires the offline calculation of an infinite horizon cost in terms of Lemma 1 and Assumption 1. For the case of symmetric constraints, in the next step, we derive LMI conditions which determine a terminal set associated with a *timeinvariant* terminal controller. Then, we formulate new LMI conditions which lead to a *parameter-varying* terminal controller.

*Lemma 2:* [17] (Fixed terminal control law) Suppose that the LPV system (1) is subject to symmetric constraints (3). If there exist a scalar  $\alpha$ , matrices X > 0 with  $X \in \mathbb{R}^{n_x \times n_x}$ , and  $Y \in \mathbb{R}^{n_y \times n_x}$  such that

$$\alpha > 0, X > 0, \tag{13a}$$

$$\begin{vmatrix} X & * & * & * \\ A_i X + B_i Y & X & * & * \\ X & 0 & \alpha Q^{-1} & * \\ Y & 0 & 0 & \alpha R^{-1} \end{vmatrix} > 0, \quad (13b)$$

$$\begin{bmatrix} y_{m,max}^2 & e_m^T (C_i X + D_i Y) \\ * & X \end{bmatrix} \ge 0,$$
(13c)  
$$m = 1, 2, \dots, n_y, \quad i = 1, 2, \dots, N,$$

where  $e_m$  is the *m*-th element of a basis vector in  $R^{n_y}$ . Then, with the fixed state feedback control law  $\kappa(x) = YX^{-1}x$  the region  $\Omega(\alpha)$  with  $P = \alpha X^{-1}$  is robustly invariant.

The time-invariant feedback law introduces unnecesssary conservativeness. It was shown in [18] that a parameter-dependent terminal control law leads to less restrictive LMI conditions.

*Lemma 3:* [18] (Parameter-varying terminal control law) Suppose that the LPV system (1) is subject to symmetric constraints (3). If there exist a scalar  $\alpha$ , matrices X > 0 with  $X \in \mathbb{R}^{n_x \times n_x}, Y_j \in \mathbb{R}^{n_y \times n_x}, j = 1, 2, \dots, N, \mathcal{T}_{ij}$  and  $\mathcal{M}_{ij}$   $(i, j = 1, 2, \dots, r)$  such that  $\alpha > 0, X > 0$ , (14a)

$$\begin{cases} \mathcal{L}_{ii} \geq \mathcal{T}_{ii}, \quad i = 1, 2, \cdots, N, \\ \mathcal{L}_{ij} + \mathcal{L}_{ji} \geq \mathcal{T}_{ij} + \mathcal{T}_{ij}^T, \\ [\mathcal{T}_{\cdot \cdot}] > 0 \quad i < i \end{cases}$$
(14b)

$$\begin{cases} \mathcal{F}_{ii,m} \geq \mathcal{M}_{ii,m}, & i = 1, 2, \cdots, N, \\ \mathcal{F}_{ij,m} + \mathcal{F}_{ji,m} \geq \mathcal{M}_{ij,m} + \mathcal{M}_{ij,m}^T, & (14c) \\ [\mathcal{M}_{ij,m}]_{N \times N} \geq 0, & j < i. \end{cases}$$

Then, the parameter-varying state feedback control  $u = \kappa(\lambda)x$  guarantees that the LPV system is robustly invariant in the region  $\Omega(\alpha)$ . In the above,  $\kappa(x) = \sum_{j=1}^{N} \theta_j(k) K_j x$  with  $K_j = Y_j X^{-1}$ ,  $V(x) = x^T P x$  with  $P = \alpha X^{-1}$  and

$$\mathcal{L}_{ij} = \begin{bmatrix} X & * & * & * \\ A_i X + B_i Y_j & X & * & * \\ X & 0 & \alpha Q^{-1} & * \\ Y_j & 0 & 0 & \alpha R^{-1} \end{bmatrix},$$
$$\mathcal{F}_{ij,m} = \begin{bmatrix} y_{m,max}^2 & e_m^T (C_i X + D_i Y_j) \\ * & X \end{bmatrix}.$$

**Remark 5:** In order to obtain the feasible region of the online optimization problem P2 as large as possible, one can solve the offline optimization problem

$$\max_{\alpha, X, Y} \max_{(\det X)^{\frac{1}{n_x}}, \quad \text{s.t.} \quad (13),$$

or

 $\underset{\alpha, X, Y_1, Y_2, \cdots, Y_N}{\text{maximize}} \quad (\det X)^{\frac{1}{n_x}}, \quad \text{s.t.} \quad (14),$ 

respectively, to get the static terminal control law or the parameter-dependent terminal control law. Both of the optimization problems are convex and can be solved by standard LMI solvers [19].

**Remark 6:** For LPV systems with asymmetric constraints, there is no systematic way to get ellipsoidal terminal regions except for choosing tighter, symmetric constraints within the asymmetric bounds.

## C. MPC for LPV systems with symmetric constraints

In this subsection we will present a more efficient algorithm for the optimization problem P2 in the case of symmetric constraints. We choose the matrix P as a new online optimization variable and transform the problem P2 to an SDP. In other words, the terminal control law, the terminal set and the terminal penalty are determined online as well.

Minimization of  $x(k/k)^T Q x(k/k) + u(k/k)^T R u(k/k) + x(k + 1/k)^T P x(k + 1/k)$ with  $P \succ 0$  is equivalent to

$$\begin{array}{ll} \underset{\alpha, \ u(k/k)}{\text{minimize}} & \alpha, & s.t. \\ x(k/k)^T Q x(k/k) + u(k/k)^T R u(k/k) \\ + x(k+1/k)^T P x(k+1/k) \leq \alpha. \end{array}$$

By the Schur complement [19], this is equivalent to

$$\begin{array}{cccc}
& \min_{\alpha, \ u(k/k)} & \alpha, & s.t. \\
& \begin{bmatrix} 1 & * & * & * \\ x(k/k) & \alpha Q^{-1} & * & * \\ u(k/k) & 0 & \alpha R^{-1} & * \\ \Delta & 0 & 0 & X \end{bmatrix} \ge 0, \quad (15)$$

with  $\Delta = A(\theta(k))x(k/k) + B(\theta(k))u(k/k)$  and  $X = \alpha P^{-1}$ . Due to  $x(k+1/k)^T Px(k+1/k) \leq \alpha$ , which follows from (15), the optimization problem P2 with parameter varying terminal control law is formulated as:

**Problem P3:** Consider the following LMI optimization problem

$$\min_{\alpha, u(k/k), X, Y_1, Y_2, \cdots, Y_N} \alpha,$$

subject to

$$\begin{split} \bar{x}(k+1/k) &= A(\theta(k))\bar{x}(k/k) + B(\theta(k))\bar{u}(k/k), \\ \bar{y}(k/k) &= C(\theta(k))\bar{x}(k/k) + D(\theta(k))\bar{u}(k/k), \\ \bar{y}(k/k) &\in \mathcal{H}_0, \\ \bar{x}(k/k) &= x(k), \\ (14), (15). \end{split}$$

*Corollary 1:* Assume that the optimization problem P3 is feasible at initial time. Then, the MPC strategy based on P3 guarantees that

- P3 is feasible for all k > 0,
- The symmetric constraints (3) are satisfied for all time instants,
- The MPC control law asymptotically stabilizes the LPV system (1).

*Proof:* Let  $\{\alpha^*, u^*(k), X^*, Y_1^*, Y_2^*, \cdots, Y_N^*\}$  denote the solution to the optimization problem P3, associated with the minimum cost  $\alpha^*(k)$ , at the sampling time k. We know from Theorem 1 that  $\{\alpha^*, K^*(k)x(k + 1/k), X^*, Y_1^*, Y_2^*, \cdots, Y_N^*\}$  is a feasible solution for the state x(k+1) at time instant k+1, where  $K^*(k) = \sum_{i=1}^N \theta_i(k)Y_i^*(X^*)^{-1}$ .

The proof of asymptotically stability and constraints satisfaction can be completed following the lines of the proof of Theorem 1.  $\Box$ 

**Remark 7:** If  $Y = Y_1 = \ldots = Y_N$ , the optimization problem P3 with parameter-varying terminal control law is replaced by a problem with static terminal control law, which has been proposed in [11, 12]. The parameter-varying terminal control law provides extra degree of freedom in the optimization problem P3, which promises a larger feasible region and better performance, however, at the cost of higher computational burden.

## IV. NUMERICAL EXAMPLE

**Example 1** In order to test the effectiveness of the proposed MPC algorithm, according to Corollary 1, we first revisit Example 2 reported in [5]. As in [7], we assume that the input constraint is  $|u(k)| \le 1$ , and  $\theta(y(k)) = 0.5 + 50y(k)^2$  with  $y(k) = x_2(k) - x_1(k)$ . Furthermore, we assume that the output constraint  $|y(k)| \le 0.5$  is imposed. We chose  $\theta_{min} = 0.5$ , and  $\theta_{max} = \theta(0.5) = 13$ .

The initial condition for the system states is given as  $x_0 = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}^T$ . The weighting matrices have been chosen,  $Q = \text{diag}\{1, 1, 1, 1\}$  and R = 1. The performance of the proposed method were compared with previous results, which are [5] with static terminal control law and [11] with self-scheduling controller. We see from Figure 1 that the algorithm P3, provides a feasible solution and the smallest upper bound on the infinite-horizon cost among the considered methods. At the same time, Figure 1 shows that the proposed MPC algorithm leads to faster convergence than the other two algorithms.

Example 2 Consider the discrete-time system

$$\begin{aligned} x_1(k+1) &= 0.5x_1^2(k) + u, \\ x_2(k+1) &= (0.5 + \sin^2(k)/3)x_1(k) + x_2(k) + u, \end{aligned}$$

with the input constraint  $-1.0 \le u \le 0.5$ , the state constraints  $-1 \le x_1, x_2 \le 1$ . The initial condition

for the system states is given as  $x_0 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ . The control objective is to regulate the initial state to the equilibrium while satisfying the constraints. It is possible to embed the nonlinear system into the prescribed polytope [19]

$$\Xi = Co\left\{ \begin{bmatrix} 1/2 & 0 & 1\\ 1/2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1/2 & 0 & 1\\ 5/6 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1/2 & 0 & 1\\ 1/2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} -1/2 & 0 & 1\\ 5/6 & 1 & 1 \end{bmatrix} \right\}.$$
(17)

Thus, we can use the proposed method according to Theorem 1. We choose the weighting matrices as  $Q = \text{diag}\{1, 1\}$  and R = 1, respectively. As discussed in Remark 2, the optimization problem P2 is a convex optimization problem since the input and state constraints sets are convex. To solve the problem, we used CVX, a package for specifying and solving convex optimization problem [20, 21]. For the example system, it is observed that the proposed method achieves good performance as well as constraint satisfaction. Furthermore, we point out that there is no feasible solution to the algorithms proposed in [5, 11], which only consider symmetric constraints. If we use these approaches to deal with asymmetric constraints, tighter constraints have to be chosen which lead to a smaller feasible region.

## V. CONCLUSIONS

In this paper we proposed an MPC scheme for LPV systems with possibly asymmetric constraints. The obtained online optimization problem is a convex optimization problem and allows the first control action to be chosen freely. We have shown that the proposed scheme guarantees recursive feasibility and closedloop stability if the optimization problem is feasible at initial time. In the case of symmetric constraints, by determining the terminal control law, terminal penalty term, and terminal region online, a less conservative MPC controller has been derived. The obtained online optimization problem is subject to LMIs. Numerical examples illustrate the effectiveness of the results.

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Fig. 1. Comparison of the proposed scheme (solid line) with the controllers [11] (dashed line) and [5] (dash-dot line).



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